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Cancellation of Linearized Axion–Dilaton Self-Action Divergence in Strings

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The force densities exerted on a localized material system by linearized interaction with fields of axionic and dilatonic type are shown to be describable very generally by relatively simple expressions that are well behaved for fields of purely external origin, but that will be subject to ultraviolet divergences requiring regularization for fields arising from self-interaction in submanifold-supported "brane"-type systems. In the particular case of a two-dimensionally supported, i.e., string-type, system in an ordinary four-dimensional background it is shown how the result of this regularization is expressible in terms of the worldsheet curvature vector K^{μ}, and more particularly that (contrary to what was suggested by early work on this subject) for a string of Nambu–Goto type the divergent contribution from the dilatonic self-action will always be directed oppositely to its axionic counterpart. This makes it possible for the dilatonic and axionic divergences entirely to cancel each other out (so that there is no need of a renormalization to get rid of "infinities") when the relevant coupling coefficents are related by the appropriate proportionality condition provided by the low-energy limit of superstring theory.

1. INTRODUCTION

This article is intended as a contribution to the clarification of a question raised by Dabholkar and Harvey (1989), who pointed out that the finiteness of superstring theory implied a similar finiteness for the corresponding low-energy classical limit theory, as constructed in terms of Nambu–Goto-type strings interacting via coupling to gravitational, dilatonic, and axionic type fields. What this means is that although the gravitational, dilatonic, and axionic self-interaction contributions will each be separately divergent in generic classical string models, their net effect should cancel out—so that no "infinite" renormalization is required—in the special case of the particular

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model obtained from the low-energy limit of superstring theory. Although there has always been a general consensus to the effect that this noteworthy prediction is indeed correct, the question of how the expected cancellation actually comes about has been the subject of considerable discord.

Relying on the straightforward—but with hindsight obviously flawed—analysis of Dabholkar and Harvey, the opinion that was most widely held until recently is succinctly summarized by the assertion (Dabholkar *et al.*, 1990) that, "Since the quantum answer is zero the classical answer must be zero. This is indeed true. There are three divergent contributions to the classical self energy. The dilaton contribution is the same as the axion but the gravitational contribution is negative and twice as large." The flaw in in the last sentence of this particular analysis is that it takes account only of the external field energy contributions, while neglecting the internal (world sheet supported) contributions, either because the authors assumed (wrongly) that they would be relatively negligible, or else because they simply forgot about them, perhaps due to having started by working out the axion part, for which the previous literature was already well developped (Vilenkin and Vachaspati, 1987) and for which it just so happens that there is no internal contribution.

There was also a noteworthy dissenting opinion: while agreeing that that total was indeed zero, it was claimed on the basis of a rather obscure calculation by Copeland *et al.* (1990) that—far from being the same as that of the axion—the dilaton contribution canceled out all by itself, leaving the axion contribution to be canceled just by the gravitational contribution. It appears with hindsight that this nonconformist conclusion was not quite so wide of the mark as that of Dabholkar and Harvey, but it does not seem to have been taken so seriously, presumably because of the failure to make clear the logical reasoning on which it was supposed to have been based.

It has now become evident that both of these competing alternatives were incorrect, essentially for the same underlying reason, which was the use of naive force or energy formulas based on the omission of various important terms that were wrongly assumed to be negligible or simply forgotten about. These overhasty omissions were originally motivated to a large extent by the technical difficulties involved in actually evaluating the terms in question, but such difficulties have recently been alleviated by the introduction of more efficient geometrical methods. Following the derivation by these methods of the correct formula for the complete gravitational force contribution (Battye and Carter, 1995) acting on a general string model, it has recently been found (Carter and Battye, 1998) that the divergent part of the gravitational self-force cancels itself out all by itself in the case of a Nambu–Goto string in four dimensions. The implication of this is that—in order for the total to vanish in the low-energy classical limit derived from superstring theory—the corresponding dilaton contribution should neither be equal to the axion contribution (as asserted by Dabholkar and Harvey) nor even zero (as asserted by Copeland *et al.*,), but in fact exactly opposite to the axion contribution, at least in a four-dimensional background.

The purpose of the present work is to provide a direct verification of this corollary, i.e., of the mutual cancellation of the relevant axionic and dilatonic string self-action divergences for the linearized classical limit derived from superstring theory in four dimensions. A less direct confirmation, and a generalization to higher dimensions (where the total still vanishes, but not the gravitational part on its own) has already been provided using a new approach based on the use of an effective action by Buonanno and Damour (1998). It is worthwhile to provide further confirmation because the pertinence of this kind of approach was explicitly, but unjustly, cast into doubt by Copeland *et al.* (1990), who alleged that "in general it is not correct" to deduce the divergent part of the self-force from the divergent part of the effective action, the reason for their scepticism being the discrepancies that arose in their own rather incoherent approach.

The verdict of the present analysis is that use of the effective action approach is inherently correct after all, and that there are no discrepancies, provided all the calculations are carried out properly without omission of relevant terms. That an approach based on a force analysis is perfectly consistent with an approach using an effective action had previously been demonstrated for electromagnetic interactions in strings (Carter, 1997b). Moreover, the the results of the effective action approach used by Buonanno and Damour (1998) were known from the outset to be consistent with the detailed force analysis—as correctly carried out with all relevant terms included—for purely gravitational interactions (Battye and Carter, 1995). In this gravitational case the detailed relationship between the two kinds of approach has since been demonstrated explicitly (Carter, 1999). The present work provides an analogously explicit demonstration that the effective action approach used by Buonanno and Damour (1989) is consistent with the detailed force analysis for the technically simpler cases of dilatonic and axionic interactions.

2. LAGRANGIAN FOR DILATONIC AND AXIAL CURRENT COUPLINGS

Before considering the divergences that arise from self-interaction it is first necessary to consider the effect of linear interactions with generic fields of the kinds with which we are concerned, namely a dilatonic (scalar) field ϕ and an axionic (pseudoscalar type) field represented by an antisymmetric Kalb–Ramond type 2-form $B_{\mu\nu}$, say. We shall ultimately be concerned with the application to the classical low-energy limit of superstring theory, which also involves a symmetric gravitational perturbation field $h_{\mu\nu}$. However, due to the assumption of linearity (which will be physically justified when the fields are sufficiently weak) each of the three pertinent (dilatonic, axionic, and gravitational) parts can be treated independently of the other two. Since the relevant analysis of the gravitational part is already available (Battye and Carter, 1995; Carter and Battye, 1998; Carter, 1999) the present article will be restricted to the corresponding analysis for the technically simple axionic and dilatonic parts. Unlike the dilatonic part, the axionic part seems to have been correctly treated in the earlier work (Dabholkar and Harvey, 1989; Copeland *et al.*, 1990), but we shall work through it again here in order to demonstrate the use of the neater, more efficient, and therefore less error-prone mathematical formalism that has since been developed (Carter, 1996, 1997a) and that is indispensible for avoiding unnecessarily heavy algebra in more complicated applications such as the gravitational part, and is helpful even for relatively simple applications such as to the dilatonic part dealt with here.

The action governing the kind of system to be analyzed here will consist of a total $\mathcal{I}_{\rm to} = \mathcal{I}_{\rm ra} + \mathcal{I}_{\rm ma}$ in which the first term is a a free radiation contribution of the form

$$\mathcal{I}_{\rm ra} = \int \hat{\mathcal{L}}_{\rm ra} \, \|g\|^{1/2} \, d^4x \tag{1}$$

where $\|g\|$ is the modulus of the determinant of the 4-dimensional spacetime background metric $g_{\mu\nu}$ as expressed with respect to local coordinates x^μ, and where $\hat{\mathcal{L}}_{\rm ra}$ is a Lagrangian density scalar that depends only—in a homogeneous quadratic manner—on the relevant linearized long-range field variables, which in the present instance are ϕ and $B_{\mu\nu}$. The other part of the action is a material contribution

$$\mathcal{I}_{\rm ma} = \int \hat{\mathcal{L}}_{\rm ma} \, \|g\|^{1/2} \, d^4x \tag{2}$$

involving another Lagrangian contribution $\hat{\mathcal{L}}_{\rm ma}$ that is restricted to have a purely linear dependence on these long-range field variables, while also having a generically nonlinear dependence on whatever other variables may be needed to characterize the localized material system under consideration, which in the application that follows will be taken to be a string. It will be postulated that the material system is unpolarized in the technical sense that its dependence on the linearized long-range field variables does not involve derivatives, which means that its Lagrangian density scalar will have the form

$$\hat{\mathscr{L}}_{\rm ma} = \hat{\mathscr{L}} + \hat{\mathscr{L}}_{\rm co} \tag{3}$$

in which the primary contribution $\hat{\mathscr{L}}$ is entirely independent of the linearized long-range field variables ϕ and $B_{\mu\nu}$, while the coupling term $\hat{\mathscr{L}}_{\rm co}$ will have the homogeneous form

$$\hat{\mathscr{L}}_{\rm co} = \frac{1}{2}\hat{W}^{\mu\nu}B_{\mu\nu} + \hat{T}\phi \tag{4}$$

in which the antisymmetric tensor field $\hat{W}^{\mu\nu}$ and the scalar field $\hat{T}$ are each independent of both $B_{\mu\nu}$ and ϕ.

Unlike the linear coupling term, the homogeneously quadratic free radiation contribution $\mathscr{L}_{\rm ra}$ will involve field gradients. This contribution will be given in terms of covariant differentiation with respect to the background metric $g_{\mu\nu}$, for which the usual symbol ∇ will be used, by an expression of the form

$$\mathscr{L}_{\rm ra} = -\frac{1}{8\pi}\left(m_{\rm D}^2\phi^{;\mu}\phi_{;\mu} + \frac{1}{6m_{\rm J}^2}J^{\mu\nu\rho}J_{\mu\nu\rho}\right) \tag{5}$$

using the notation

$$\phi_{;\mu} = \nabla_\mu\phi, \qquad J_{\mu\nu\rho} = 3\nabla_{[\mu}B_{\nu\rho]} \tag{6}$$

where square brackets indicate index antisymmetrization. The parameters $m_{\rm D}$ and $m_{\rm J}$ in this expression are fixed coupling constants having the dimensions of mass on the understanding that we are using units in which the speed of light c and the Dirac–Planck constant $\hbar$ are set to unity. The quantity $m_{\rm D}$ is what may conveniently be referred to as the Dicke mass. This dilation mass scale should not be confused with the dilaton mass, m_ϕ say, which is usually assumed to be very small, and which is simply taken to be zero in the present work. The Dicke dilation mass scale $m_{\rm D}$ is usually supposed to be very large, at least comparable with the Planck mass defined by $m_{\rm P} = G^{-1/2}$, where G is Newton's constant. In particular, if the theory is to be applied to the modern solar system, then there are severe observational limits (Dicke, 1964) that can be interpreted as implying that the relevant value of the dimensionless Brans–Dicke parameter $\omega = 2Gm_{\rm D}^2 + 3/2$ should be very large compared with unity and hence that $m_{\rm D}$ is large compared with $m_{\rm P}$.

The other mass scale, $m_{\rm J}$, is what may conveniently be referred to as the Joukowski mass, since the corresponding Kalb–Ramond coupling—of the axial kind associated in the context of superstring theory with the names of Wess and Zumino—gives rise to a lift force of the type that has long been well known in the context of aerofoil theory, where it was originally derived as a corollary of the Magnus effect by the Russian theoretician Joukowski.

This mass scale can also usually be supposed to be very large, unlike the associated axion mass, $m_{\rm a}$ say, with which it should not be confused. Like the dilaton mass m_ϕ, the axion mass $m_{\rm a}$ is usually assumed to be very small, and will be taken to be zero for the purposes of the present work. The axial current model obtained in this way is interpretable as the "stiff" (Zel'dovich type) limit (characterized by sound perturbation propagation at the speed of light) within the more general category of ordinary perfect fluid models. The Kalb–Ramond Wess–Zumino Joukowski coupling bivector field $\hat{W}^{\mu\nu}$ is interpretable (Carter, 1994) as a vorticity flux, and must be such as to satisfy a flux conservation law of the form

$$\nabla_\mu \hat{W}^{\mu\nu} = 0 \tag{7}$$

in order to ensure invariance under local Kalb–Ramond gauge transformations of the form $B_{\mu\nu} \mapsto B_{\mu\nu} + 2\nabla_{[\mu}\varphi_{\nu]}$ for an arbitrary covector field φ_ν. In typical applications to continuous media (Carter, 1994) this condition is fullfilled by by a specification of the form $\hat{W}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\,\chi^+_{;\rho}\,\chi^-_{;\sigma}$, where the scalars χ^+ and χ^- are two of the intrinsic field variables characterizing the material system and $\epsilon^{\mu\nu\rho\sigma}$ is the usual antisymmetric measure tensor induced on the background space (modulo a choice of signature) by the metric $g_{\mu\nu}$. In such a case, as in the case of the string-type systems for which the relevant formula for $\hat{W}^{\mu\nu}$ will be described below, the corresponding coupling action has the special feature—which it shares with its electromagnetic analogue (Carter, 1997b)—that its dependence on the metric $g_{\mu\nu}$ cancels out. Failure to make proper allowance for the absence of this familiar simplifying property in more general couplings, and in particular in couplings of gravitational and gravitational type, seems to be one of the main reasons why early evaluations of the latter (Dabholkar and Harvey, 1989: Copeland *et al.*, 1990) were so systematically erroneous.

The conditions on the other coupling term are more restrictive. In principle the scalar source coefficient $\hat{T}$ might have various forms for diverse scalar coupling theories that might be conceived, but in order for the coupling to be describable as "dilatonic" in the sense associated most particularly with the work of Dicke (1964), it must be given by the trace

$$\hat{T} = \hat{T}_\mu{}^\mu \tag{8}$$

of the material stress-energy tensor that is obtained from the primary Lagrangian contribution $\mathcal{L}$ of the system according to the usual geometric specification

$$\hat{T}^{\mu\nu} = 2\|g\|^{-1/2}\frac{\partial(\hat{\mathcal{L}}\|g\|^{1/2})}{\partial g_{\mu\nu}} = 2\frac{\partial\hat{\mathcal{L}}}{\partial g_{\mu\nu}} + \hat{\mathcal{L}}g^{\mu\nu} \tag{9}$$

The reasoning whereby Dicke and other early workers (notably his predecessor Jordan and his colleague Brans) were lead to a coupling of this kind was based on the postulate that in a fully nonlinear treatment the primary material action contribution would be given by expression of the form $\mathcal{I}_{\mathrm{D}} = \int \mathcal{L}_{\mathrm{D}} \, \|g_{\mathrm{D}}\|^{1/2} \, d^4 x$, in which the Lagrangian density $\hat{\mathcal{L}}_{\mathrm{D}}$ depends on whatever intrinsic material constituent fields may be necessary as well as on a certain background metric $g_{\mathrm{D}\mu\omega}$, say, that may conveniently be referred to as the Dicke metric, in order to distinguish it from the associated Einsten metric, $g_{\mathrm{E}\mu\nu}$ say. The latter is characterized by the condition that the relevant gravitational field action be proportional to the spacetime volume integral of its Ricci tensor. In a fully nonlinear treatment the Dicke metric is related to the Einstein metric by a conformal transformation of the form $g_{\mathrm{D}\mu\nu} = e^{2\phi} \, g_{\mathrm{E}\mu\nu}$. In a linearized treatment such as is used here, the relevant Einstein metric can be taken to have the form $g_{\mathrm{E}\mu\nu} \simeq g_{\mu\nu} + h_{\mu\nu}$ where the "unperturbed" background metric $g_{\mu\nu}$ is strictly flat or at least Ricci flat, and where $h_{\mu\nu}$ is the usual gravitational perturbation tensor. The linearization implies that one can also take $e^{2\phi} \simeq 1 + 2\phi$ and hence that the relevant Dicke metric can be taken to be given in terms of the flat or Ricci flat background by a relation of the form $g_{\mathrm{D}\mu\nu} \simeq g_{\mu\nu} + h_{\mu\nu} + 2\phi g_{\mu\nu}$.

In the traditional kind of dilaton coupling theory theory—as envisaged by Dicke (1964)—and also in the particular kind of low-energy classical limit of superstring theory that was considered by Dabholkar and Harvey (1989), the ϕ dependence only comes in indirectly via the dependence on $g_{\mathrm{D}\mu\nu}$. [Rather more complicated couplings occur in some of the more eleborate models derived from superstring theory or M-theory, but, as remarked by Cho and Keum (1998), this does not necessarily affect the form of their linearized weak-field limits.] Under such conditions it follows that in the linearized limit we shall simply have $\mathcal{L}_{\mathrm{D}}\|g_{\mathrm{D}}\|^{1/2} \simeq (\hat{\mathcal{L}} + 1/2 \; \hat{T}^{\mu\nu}(h_{\mu\nu} + 2\phi g_{\mu\nu}))\|g\|^{1/2}$, where $\hat{\mathcal{L}}$ is obtained by substituting the unperturbed background metric $g_{\mu\nu}$ in place of the Dicke metric $g_{\mathrm{D}\mu\nu}$ in $\mathcal{L}_{\mathrm{D}}$ and $\hat{T}^{\mu\nu}$ is the associated stress-energy tensor as given by the usual formula (9). The consequences of the purely gravitational part of the coupling that arises in this way have already been treated elsewhere (Battye and Carter, 1995; Carter and Battye, 1998; Carter, 1998). The present work will be concerned just with the dilatonic part, which will evidently be specified by a coupling term that reduces to the simple form $\hat{T}\phi$ as presented in (4) with $\hat{T}$ as given by (8) and (9).

Not much can be said about the equations of motion for the internal fields characterizing the material system until the form of its primary Lagrangian contribution $\hat{\mathcal{L}}$ has been specified, but it is evident quite generally that independently of such details, in an empty background the equation of motion for the axion field will be obtainable, using the gauge condition

$$\nabla_{\mu} B_{\mu\nu} = 0 \tag{10}$$

(the analogue of the Lorentz condition familiar in the context of electromagnetism) in the well-known (Vachaspati and Vilenkin, 1987) D'Alembertian form

$$\nabla_\sigma \nabla^\sigma B_{\mu\nu} = -4\pi m_{\mathrm{J}}^2 \hat{W}_{\mu\nu} \tag{11}$$

For the dilaton field no question of gauge arises at this stage: the relevant field equation is thus immediately obtainable in the simple form

$$\nabla_\sigma \nabla^\sigma \phi = -4\pi m_{\mathrm{D}}^{-2} \hat{T} \tag{12}$$

3. DISTRIBUTIONAL SOURCES

As in the more familiar case of point-particle models, or "zero-branes," the problem of ultraviolet divergences will arise for higher dimensional "p-branes," and in particular for string models, with $p = 1$, because in these cases the relevant source densities $\hat{W}^{\mu\nu}$ and $\hat{T}^{\mu\nu}$ are not regular functions, but Dirac-type distributions that vanish outside the relevant one- or two-dimensional worldsheets, except of course in the case of an ordinary medium, with the maximal space dimension, namely $p = 3$, in an ordinary 4-dimensional spacetime background of the kind to which the present analysis is restricted.

In the case of a general p-brane, with local $(p + 1)$-dimensional worldsheet embedding given by $x^\mu = \bar{x}^\mu\{\sigma\}$ in terms of intrinsic coordinates σ^i $(i = 0, 1, \ldots, p)$, so that the induced surface metric will have the form

$$\gamma_{ij} = g_{\mu\nu} \bar{x}^\mu_{,i} \bar{x}^\nu_{,j} \tag{13}$$

the relevant source distributions will be expressible using the terminology of Dirac delta "functions" in the form exemplified in the case of the vorticity flux by

$$\hat{W}^{\mu\nu} = \|g\|^{-1/2} \int \overline{W}^{\mu\nu}\, \delta^4[x - \bar{x}\{\sigma\}] \|\gamma\|^{1/2}\, d^{p+1}\sigma \tag{14}$$

where $\|\gamma\|$ is the determinant of the induced metric (13), and where the surface vorticity flux bivector $\overline{W}^{\mu\nu}$ is a *regular* antisymmetric tensor field on the worldsheet (but undefined off it). The analogous expression for the stress-momentum-energy source will be given by

$$\hat{T}^{\mu\nu} = \|g\|^{-1/2} \int \overline{T}^{\mu\nu}\, \delta^4[x - \bar{x}\{\sigma\}] \|\gamma\|^{1/2}\, d^{p+1}\sigma \tag{15}$$

where the surface stress-momentum-energy density $\overline{T}^{\mu\nu}$ is a *regular* symmetric tensor field on the worldsheet (but undefined off it).

The distributional nature of these source terms, in cases for which $p < 3$, is a corollary of the similarly distributional nature of the action density $\hat{\mathcal{L}}_{\rm ma}$ as defined, according to (3), to include both the purely internal contribution $\hat{\mathcal{L}}$ and the cross-coupling contribution $\hat{\mathcal{L}}_{\rm co}$. This distributional action density is expressible in the form

$$\hat{\mathcal{L}}_{\rm ma}\, \|g\|^{-1/2} \int \overline{\mathcal{L}}_{\rm ma}\, \delta^4[x - \overline{x}\{\sigma\}]\|\gamma\|^{1/2}\, d^{p+1}\sigma \tag{16}$$

with

$$\overline{\mathcal{L}}_{\rm ma} = \overline{\mathcal{L}} + \overline{\mathcal{L}}_{\rm co} \tag{17}$$

where the primary contribution or "master function" $\overline{\mathcal{L}}$ and the secondary coupling contribution $\overline{\mathcal{L}}_{\rm co}$ are both well-behaved scalar functions on the string worldsheet, but undefined off it. Of these, the master function $\overline{\mathcal{L}}$ will be the intrinsic worldsheet Lagrangian, defined as a function just of the relevant internal, worldsheet-confined, fields, such as currents, on the string and of its induced metric, while the cross-coupling contribution will be given in terms of the worldsheet-confined fields $\overline{W}^{\mu\nu}$ and $\overline{T}^{\mu\nu}$ by

$$\overline{\mathcal{L}}_{\rm co} = \tfrac{1}{2}\,\overline{W}^{\mu\nu}B_{\mu\nu} + \overline{T}\phi \tag{18}$$

where $\overline{T} = \overline{T}^{\mu}_{\mu}$.

In terms of these well-behaved worldsheet functions, the corresponding material contribution (2) to the action can be expressed directly, without recourse to heavy distributional machinery, as a simple $(p + 1)$-surface integral in the form

$$\mathcal{I}_{\rm ma} = \int \overline{\mathcal{L}}_{\rm ma}\|\gamma\|^{1/2}\, d^{p+1}\sigma \tag{19}$$

In particular, the regular surface stress-energy tensor $\overline{T}^{\mu\nu}$ needed for the purpose of applying (18) is obtainable directly from the worldsheet master function, without the use of distributions, using the formula

$$\overline{T}^{\mu\nu} = 2\|\gamma\|^{-1/2}\,\frac{\partial(\overline{\mathcal{L}}\|\gamma\|^{1/2})}{\partial g_{\mu\nu}} \tag{20}$$

As was remarked above, the errors in the early literature on this subject (Dabholkar and Harvey, 1989; Copeland *et al.*, 1990) were largely attributable to failure to take proper account of the fact that unlike what occurs in the historically familiar special cases of electromagnetic and axionic coupling, for more general cases such as those of gravitational and dilatonic coupling the relevant coupling action contribution will be metric dependent. In order to allow for this, it is useful to work out the appropriately constructed hyper-

Cauchy tensor (a relativistic generalization of the Cauchy elasticity tensor of classical mechanics), which is defined by

$$\overline{\mathcal{C}}^{\mu\nu\rho\sigma} = \|\gamma\|^{-1/2} \frac{\partial}{\partial g_{\mu\nu}} (\overline{T}^{\rho\sigma} \|\gamma\|^{1/2}) \tag{21}$$

or equivalently, in manifestly symmetric form, by

$$\overline{\mathcal{C}}^{\mu\nu\rho\sigma} = \overline{\mathcal{C}}^{\rho\sigma\mu\nu} = 2\|\gamma\|^{-1/2} \frac{\partial^2(\overline{\mathcal{L}}\, \|\gamma\|^{1/2})}{\partial g_{\mu\nu}\, \partial g_{\rho\sigma}} \tag{22}$$

4. BRANE WORLDSHEET GEOMETRY

If the relevant radiation fields $B_{\mu\nu}$ and ϕ are considered to be regular background fields attributable to external souces, the treatment of a brane system of the kind described in the preceding section will be straightforward, but it is evident that this will not be the case for the radiation fields produced by the brane itself, since they will be singular just where their evaluation is needed.

Even for the treatment just of the regular case in which the relevant radiation fields are of purely external origin, and *a fortiori* for the treatment of the more delicate problem of self-interaction, it is desirable—before proceeding to the derivation of the dynamical equations that ensue from the action (19)—to recapitulate the essential geometric concepts (Carter, 1992; Carter, 1997a) that are needed for the kinematic description of the evolving worldsheet. The unavailability of this machinery at the time of the pioneering work (Dabholkar and Harvey, 1989; Copeland *et al.*, 1990) on the Goto–Nambu case is one of the reasons for the use of the misleading shortcut methods responsible for the confusion that beset this subjet before the recent clarification (Carter and Battye, 1998; Buonanno and Damour, 1998.

Point-particle kinematics can conveniently be developed in terms of the timelike worldline tangent vector u^μ that is uniquely fixed by the condition of being future-directed with unit normalization $u^\mu u_\nu = -1$, and of the associated acceleration vector that is given in terms of covariant differentiation with respect to the (flat or curved) spacetime background metric $g_{\mu\nu}$ by $a^\mu = u^\nu \nabla_\nu u^\mu$. For higher brane cases, and in particular for the strings with which we shall be concerned here, a less specialized kinematic description must be used. Instead of the unique tangent vector u^μ and the derived vector a_μ that suffice for the "zero-brane" case, the kinematic behavior of higher "branes" starting with the case of strings (i.e., "one-branes") is most conveniently describable (Carter, 1992; Carter, 1997a) in terms of its first and second fundamental tensors. The former is definable as tangential projection tensor η^μ_ν, say, which is obtained by index lowering from the spacetime background projection of the inverse of the induced metric as given by the formula

$$\eta^{\mu\nu} = \gamma^{ij}\,\overline{x}^{\mu}_{,i}\,\overline{x}^{\mu}_{,j} \tag{23}$$

This *first fundamental* tensor can conveniently be used to rewrite the expressions (20) and (21) in the more practical forms

$$\overline{T}^{\mu\nu} = 2\,\frac{\partial\overline{\mathscr{L}}}{\partial g_{\mu\nu}} + \overline{\mathscr{L}}\eta^{\mu\nu} \tag{24}$$

and

$$\overline{\mathscr{C}}^{\mu\nu\rho\sigma} = \frac{\partial\overline{T}^{\rho\sigma}}{\partial g_{\mu\nu}} + \frac{1}{2}\,\overline{T}^{\rho\sigma}\eta^{\mu\nu} \tag{25}$$

The corresponding *second fundamental tensor* $K^{\rho}_{\mu\nu} = K^{\rho}_{\nu\mu}$ is obtained from the first fundamental tensor using the tangentially projected differentiation operator

$$\overline{\nabla}_{\mu} = \eta_{\mu}{}^{\nu}\,\nabla_{\nu} \tag{26}$$

according to the prescription

$$K_{\mu\nu}{}^{\rho} = \eta^{\sigma}{}_{\nu}\,\overline{\nabla}_{\mu}\eta^{\rho}{}_{\sigma} \tag{27}$$

The condition of integrability of the worldsheet is the Weingarten identity, to the effect that this second fundamental tensor should be symmetric on its first two indices, i.e.,

$$K_{[\mu\nu]}{}^{\rho} = 0 \tag{28}$$

This tensor has the noteworthy property of being worldsheet orthogonal on its last index, but tangential on its (by the Weingarten identity interchangeable) first pair of indices,

$$K_{\mu\nu}{}^{\sigma}\,\eta_{\sigma}{}^{\rho} = 0 = \perp^{\lambda}{}_{\mu}K_{\lambda\nu}{}^{\rho} \tag{29}$$

using the notation

$$\perp^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} - \eta^{\mu}{}_{\nu} \tag{30}$$

for the tensor of projection orthogonal to the worldsheet.

Whereas the full second fundamental tensor is needed for dealing with gravitational coupling (Battye and Carter, 1995; Carter and Battye, 1998), the treatment of the simpler cases considered here requires only its trace, namely the curvature vector

$$K^{\rho} = K_{\mu}{}^{\mu\rho} = \overline{\nabla}_{\nu}\eta^{\nu\rho} \tag{31}$$

which must evidently be worldsheet orthogonal, i.e.,

$$\eta^{\rho}{}_{\sigma} K^{\sigma} = 0 \tag{32}$$

In terms of the background Riemann Christoffel connection $\Gamma_{\mu}{}^{\nu}{}_{\rho} = g^{\nu\sigma}(g_{\sigma(\mu,\rho)} - 1/2\ g_{\mu\rho,\sigma)})$ this curvature vector will be expressible in explicit detail as

$$K^{\nu} = \|\gamma\|^{-1/2} (\|\gamma\|^{1/2}\ \gamma^{ij}\ \overline{x}^{\nu}_{,i})_{,j} + \gamma^{ij}\ \overline{x}^{\mu}_{,i}\ \overline{x}^{\rho}_{,j}\ \Gamma_{\mu}{}^{\nu}{}_{\rho} \tag{33}$$

In the particularly simple case of a Dirac membrane or a Nambu–Goto string (i.e., one for which the master function $\overline{\mathcal{L}}$ is just a constant) that is free, in the sense that it is not subjected to any external force, the "on-shell" configurations (i.e., the solutions of the variational dynamical equations) will simply be characterized by the condition that the vector (33) vanishes, $K^{\mu} = 0$, but this simple vanishing curvature condition will not be satisfied for more general models—such as those needed for superconducting strings (Carter, 1989; Carter and Peter, 1995; Gangui *et al.*, 1998)—nor when dilatonic and axionic forces are involved as in the cases analyzed in the present work.

It is to be remarked that the orthogonality property (32) of the curvature vector is to contrasted with the tangentiality property of the stress-energy tensor.

$$\perp^{\lambda}{}_{\mu} \overline{T}^{\mu\nu} = 0 \tag{34}$$

and of the hyper-Cauchy tensor.

$$\perp^{\lambda}{}_{\mu} \overline{\mathcal{C}}^{\mu\nu\rho\sigma} = 0 \tag{35}$$

Subject to the requirement that the worldsheet-supported field $\overline{W}^{\mu\nu}$ be constructed from internal worldsheet fields in such a way as to aquire the corresponding tangentiality property

$$\perp^{\lambda}{}_{\mu} \overline{W}^{\mu\nu} = 0 \tag{36}$$

it can be seen that the corresponding distributional conservation law (7) can be equivalently expressed in terms of tangentially projected differentiation as the regular worldsheet flux conservation law

$$\overline{\nabla}_{\mu} \overline{W}^{\mu\nu} = 0 \tag{37}$$

5. THE FORCE DENSITY FORMULA

For the purpose of the deriving the equations of motion of the material system from the variation principle, the most general variations to be considered are perturbations of the relevant internal fields, which have not yet been specified, and infinitesimal displacements with respect to the background characterized by the metric $g_{\mu\nu}$ and the the linearly coupled axionic and dilatonic fields.

The effect of displacements can be conveniently analyzed using a Lagrangian treatment in which not just the internal coordinates σ^i, but also the background coordinates x^μ are considered to be dragged along by the displacement, so that the relevant field variations are given just by the corresponding Lie derivatives with respect to the displacement vector field ξ^μ under consideration. This leads to the formulas

$$\delta B_{\mu\nu} = \xi^\sigma \nabla_\sigma B_{\mu\nu} - 2B_{\sigma[\mu}\nabla_{\nu]}\xi^\sigma \tag{38}$$

for the axionic field and

$$\delta\phi = \xi^\sigma \nabla_\sigma \phi \tag{39}$$

for the dilatonic field, while finally for the background metric itself one has the well-known formula

$$\delta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} \tag{40}$$

The postulate that the internal field equations are satisfied means that perturbations of the relevant internal fields have no effect on the action integral $\mathcal{I}_{\text{ma}}$, with the implication that for the purpose of evaluating the variation $\delta\mathcal{I}_{\text{ma}}$ there will be no loss of generality taking the variations of these so-far-unspecified internal fields simply to be zero. This means that the only contribution from the first term in (17) will be the one provided by the background metric variation, for which we obtain the familiar formula

$$\delta(\|\gamma\|^{1/2}\overline{\mathcal{L}}) = \tfrac{1}{2}\|\gamma\|^{1/2}\overline{T}^{\mu\nu}\,\delta g_{\mu\nu} \tag{41}$$

The worldsheet flux conservation law (37) is interpretable (Carter, 1992) as meaning that $\overline{W}^{\mu\nu}$ is related by Hodge-type duality to the exterior derivatives of corresponding worldsheet differential forms (which will generically be of order $p-2$, respectively). In all the usual applications (including the continuous medium example mentioned in the preceding section and the string case developed below) these differential forms will be included among (or depend only on) the relevant independently variable internal fields whose variation can be taken to be zero for the purpose of evaluationg $\delta\mathcal{I}_{\text{ma}}$ when the internal field equations are satisfied. This means that the variation of the bivector surface density will also vanish, i.e., we shall have

$$\delta(\|\gamma\|^{1/2}\,\overline{W}^{\mu\nu}) = 0 \tag{42}$$

It follows that the axionic contribution from (18) to the variation of the integrand in (19) will be given simply by

$$\delta(\|\gamma\|^{1/2}(\tfrac{1}{2}\,\overline{W}^{\mu\nu}B_{\mu\nu})) = \|\gamma\|^{1/2}(\tfrac{1}{2}\,\overline{W}^{\mu\nu}\delta B_{\mu\nu}) \tag{43}$$

The systematic absence of any contribution from the background metric variation $\delta g_{\mu\nu}$ to action variation terms such as this, not only in the axionic case considered here, but also in its more widely familiar electromagnetic analogue, sets a potentially misleading precedent that encourages a dangerous tendency—one of the main sources of error in earlier work (Dabholkar and Harvey, 1989; Copeland *et al.*, 1990)—to forget to check the possibility of metric variations in more general contexts. Although it does not contribute to the axionic term (43), allowance for the background metric variation (40) turns out to be of paramount importance not only in the gravitational case (Battye and Carter, 1995; Carter and Battye, 1998), but also for the evaluation of the dilatonic contribution with which we are concerned here. It can be seen from (25) that we shall have

$$\delta(\|\gamma\|^{1/2}\,\overline{T}) = \|\gamma\|^{1/2}\,(\overline{T}^{\mu\nu} + \overline{\mathcal{C}}^{\mu\nu})\,\delta g_{\mu\nu} \tag{44}$$

using the notation

$$\overline{\mathcal{C}}^{\mu\nu} = \overline{\mathcal{C}}^{\mu\nu\rho}{}_{\rho} \tag{45}$$

for the trace of the hyper-Cauchy tensor. Thus, despite its deceptively simple scalar nature, the dilatonic coupling gives rise to a corresponding contribution that works out to be given by an expression of the not quite so trivially obvious form

$$\delta(\|\gamma\|^{1/2}\overline{T}\phi) = \|\gamma\|^{1/2}(\overline{T}\delta\phi + (\overline{T}^{\mu\nu} + \overline{\mathcal{C}}^{\mu\nu})\phi\,\delta g_{\mu\nu}) \tag{46}$$

To evaluate the integrated effect of the contributions (41), (43), and (46), the next step is the routine procedure of substitution of the relevant Lie derivative formulas (38)–(40), followed by absorption of the terms involving derivatives of the displacement fields into pure worldsheet current divergences. For the primary contribution given by (41) one thereby obtains an expression of the familiar form

$$\tfrac{1}{2}\,\overline{T}^{\mu\nu}\,\delta g_{\mu\nu} = -\xi^{\mu}\,\overline{\nabla}_{\nu}\overline{T}^{\nu}{}_{\mu} + \overline{\nabla}_{\mu}(\xi^{\nu}\overline{T}^{\mu}_{\nu}) \tag{47}$$

while the corresponding expression for axionic coupling contribution (43) will simply be given by

$$\tfrac{1}{2}\,\overline{W}^{\mu\nu}\,\delta B_{\mu\nu} = \tfrac{1}{2}\,\xi^{\mu}N_{\mu\nu\rho}\overline{W}^{\nu\rho} + \overline{\nabla}_{\mu}(\xi^{\nu}B_{\nu\rho}\overline{W}^{\mu\rho}) \tag{48}$$

However, the dilatonic coupling contribution (46) is not so simple: in addition to the obvious scalar field variation contribution given by

$$\overline{T}\delta\phi = \xi^{\mu}\overline{T}\,\nabla_{\mu}\phi \tag{49}$$

there will be another, less obvious contribution (the one that tended to be overlooked in earlier work) given by

$$(\overline{T}^{\mu\nu} + \overline{\mathcal{C}}^{\mu\nu})\phi\delta g_{\mu\nu} = -\xi^{\mu}\overline{\nabla}_{\nu}(2(\overline{T}^{\nu}{}_{\mu} + \overline{\mathcal{C}}^{\nu}{}_{\mu})\phi) + \overline{\nabla}_{\mu}(2\xi^{\nu}(\overline{T}^{\mu}{}_{\nu} + \overline{\mathcal{C}}^{\mu}{}_{\nu})\phi) \tag{50}$$

When these expressions are used to evaluate the variation of the action integral (19) due to a displacement confined to a bounded region of the worldsheet, the application of the relevant $(p + 1)$-dimensional version of Green's theorem removes the contributions from the divergence terms, so that one is left with an expression of the standard form

$$\delta\mathcal{I}_{\rm ma} = \int \xi^{\mu}(\bar{f}_{\mu} - \overline{\nabla}_{\nu}\overline{T}^{\nu}{}_{\mu})\|\gamma\|^{1/2}\, d^{p+1}\sigma \tag{51}$$

Application of the variation principle thus gives the corresponding dynamical equation in the standard form

$$\overline{\nabla}_{\nu}\overline{T}^{\mu\nu} = \bar{f}^{\mu} \tag{52}$$

in which the vector $\bar{f}^{\mu}$ represents the total force density exerted by the various radiation fields involved. Using the foregoing expressions, we can write this force density immediately in the form

$$\bar{f}^{\mu} = \bar{f}^{\mu}_{\rm J} + \bar{f}^{\mu}_{\rm D} \tag{53}$$

in which the axionic contribution can be seen from (47) to be given by the well-known formula (the axionic analogue of the Lorentz force formula in electromagnetism)

$$\bar{f}^{\mu}_{\rm J} = \tfrac{1}{2} N^{\mu}{}_{\nu\rho}\overline{W}^{\nu\rho} \tag{54}$$

which seemed unfamiliar (Vachaspati and Vilenkin, 1987) when first derived in the present context, but which is in fact interpretable just as the immediate relativistic generalization of the historic formula on which the theory of flying is based, namely the Joukowski force law for the lift (due to the Magnus effect) on a long, thin (i.e., stringlike) aeroplane wing. What is not so well known is the corresponding formula for the dilatonic force density, which can be seen from (49) and (50) to be given by

$$\bar{f}^{\mu}_{\rm D} = \overline{T}\nabla^{\mu}\phi - \overline{\nabla}_{\nu}(2(\overline{T}^{\mu\nu} + \overline{\mathcal{C}}^{\mu\nu})\phi) \tag{55}$$

5.1. The Force on a Nambu–Goto String

The preceding formulas apply to domain wall-type membrane models (with $p = 2$) as well as to simple point-particle models (with $p = 0$), but from this stage onward we shall restrict our attention to the case of string models, as characterized by $p = 1$. Before further restricting attention to the very special case of Nambu–Goto-type string models, it is worthwhile to recapitulate some relevant features that are shared by more general string models, including the kind needed (Carter, 1989; Carter and Peter, 1994; Gangui, *et al.*, 1998) for describing the effects of the type of supercontivity whose likely occurrence in cosmic strings was originally predicted by Witten (1985).

In the higher dimensional branes the vorticity flux $\overline{W}^{\mu\nu}$ might depend on internal field variables of the model, while for a point-particle model no such source can exist at all. In the intermediate case of a string there is no obstruction to the existence of a vorticity flux, but it cannot depend on any internal field variables of the model: the only way the conservation law (37) can be satisfied on a two-dimensional worldsheet is for the vorticity flux to have the form

$$\overline{W}^{\mu\nu} = \bar{\kappa}\mathcal{E}^{\mu\nu} \tag{56}$$

where $\mathcal{E}^{\mu\nu}$ is the antisymmetric unit surface element tensor and $\bar{\kappa}$ is a constant that is interpretable as representing the momentum circulation around the string of the Zel'dovich-type fluid representing the axion field—which means that it will be an integral multiple of Planck's constant, i.e., an integral multiple of 2π in the unit system we are using with $\hbar$ set to unity. Using the traditional dot and dash notation $\dot{x}^\mu = \overline{x}^\mu_{;0}$ and $x'\mu = \overline{x}^\mu_{;1}$ for the effect of partial differentiation with respect to worldsheet coordinates σ^0 and σ^1, the antisymmetric unit surface element tensor is given by

$$\mathcal{E}^{\mu\nu} = 2(\|\gamma\|)^{-1/2}\dot{x}^{[\mu}x'^{\nu]} \tag{57}$$

In the case of a string, the fundamental tensor will be given in terms of this unit tangent bivector by

$$\eta^\mu_\nu = \mathcal{E}^\mu_\rho\mathcal{E}^\rho_\nu \tag{58}$$

One of the reasons why the kind of tensorial analysis used here was not developed much earlier for the purpose of application to string dynamics is that the heavy algebra involved in the use of coordinate-dependent expressions such as that on the right-hand side of the curvature formula (33) could be alleviated to some extent by the use of specialized internal coordinate systems of the conformal type characterized by the conditions

$$\dot{x}^\mu\, x'_\mu = 0, \qquad \dot{x}^\mu\dot{x}_\mu + x'\mu x'_\mu = 0 \tag{59}$$

which imply the relation

$$\|\gamma\|^{1/2} = x'\mu x'_\mu = -\dot{x}^\mu\,\dot{x}_\mu \tag{60}$$

If one is willing to accept the loss of flexibility entailed in restricting σ^0 and σ^1 to satisfy these conditions (which are frequently incompatible with other compelling desiderata), then the unwieldy formula (33) for the curvature vector that plays such an important role will be replaceable by the handier expression

$$K^{\nu} = \|\gamma\|^{-1/2}(x''^{\nu} - \ddot{x}^{\nu} + (x'^{\mu}x'^{\rho} - \dot{x}^{\mu}\dot{x}^{\rho})\Gamma_{\mu}{}^{\nu}{}_{\rho}) \qquad (61)$$

If the background metric $g_{\mu\nu}$ is not just empty, but actually flat, then this formula will be further simplifiable by elimination of the final term if one is willing to restrict the background cordinates to be of Minkowski type, but of couse such a restriction may not be what is most convenient for exploiting symmetries, such as occur in circular string loop configurations for which spherical or cylindrical coordinates might be preferable. The development of string dynamics has been unnecessarily delayed by overreliance on the special gauge characterized by Minkowski coordinates on the background and conformal coordinates on the worldsheet, rather that using the kind of geometrical approach followed here, which provides more elegant and concise formulas for general purposes. This more powerful geometric approach is of course particularly advantageous for applications in which for various technical reasons the usual (conformal cum Minkowski) kind of gauge may be unsuitable.

From this point on, attention will be restricted to the especially simple case of string models of Nambu–Goto type, which includes the case that arises in the low-energy limit of string theory considered by Dabholkar and Harvey (1989). Such models are characterized by the condition that the relevant master function $\overline{\mathcal{L}}$ is simply a constant, which means that it will be expressible in the form

$$\overline{\mathcal{L}} = -m_{\mathrm{K}}^2 \qquad (62)$$

where m_{K} is a fixed mass scale that will be referred to as the Kibble mass to distinguish it from other mass scales in the theory. In the context of cosmic string theory it is generally expected that it should be of the same order of magnitude as the Higgs mass, m_{X} say, that is associated with the underlying "spontaneously broken" symmetry of the vacuum. In the context of superstring theory the quantity m_{K} is usually supposed to be of the order of magnitude of the Planck mass m_{P}.

The other parameters needed to complete the specification of the theory are the quantities m_{J}, m_{D}, and $\overline{\kappa}$ that have already been introduced. To match the present formulation to the equivalent low-energy linearized limit theory in the slightly different notation used by Buonanno and Damour (1989), their parameters α, λ, and μ are identifiable as being given by the relations $\alpha = m_{\mathrm{P}}/m_{\mathrm{D}}$, $\lambda = \overline{\kappa} m_{\mathrm{P}}\, m_{\mathrm{J}}/2$, and $\mu = m_{\mathrm{K}}^2$. The special values corresponding to the low-energy superstring theory limit considered by Dabholkar and Harvey

(1989) are given by $\alpha = 1$ and $\lambda = \mu$, which in the formulation used here is equivalent to the conditions

$$m_{\rm D} = m_{\rm P}, \qquad 2m_{\rm K}^2 = \bar{\kappa} m_{\rm J} m_{\rm P} \tag{63}$$

Whether or not the particular conditions (63) are satisfied [and of course quite independently of whether the internal coordinate gauge satisfies the conditions (59) on which the specialized formulas (60) and (61) depend], the surface stress-momentum-energy tensor of the string can be seen from (23) to be proportional to the fundamental tensor, according to the formula

$$\overline{T}^{\mu\nu} = -m_{\rm K}^2 \eta^{\mu\nu} \tag{64}$$

and so its trace is given by

$$\overline{T} = -2m_{\rm K}^2 \tag{65}$$

The corresponding the hyper-Cauchy tensor is obtainable (Battye and Carter, 1995) from (25) in the form

$$\overline{\mathscr{C}}^{\mu\nu\rho\sigma} = m_{\rm K}^2(\eta^{\mu(\rho}\eta^{\sigma)\nu} - \tfrac{1}{2}\,\eta^{\mu\nu}\eta^{\rho\sigma}) \tag{66}$$

It can be seen from this that in this special Nambu–Goto case the trace tensor that appears in the dilatonic force formula (54) will vanish, i.e., one obtains

$$\overline{\mathscr{C}}^{\mu\nu} = 0 \tag{67}$$

It is to be emphasized that that this simplification is a special feature of the string case. and that it does not hold in the higher dimensional case of a Dirac membrane, nor even in the trivial lower dimensional case of a point particle with mass m and unit 4-velocity vector u^μ, for which one obtains $\overline{\mathscr{C}}^{\mu\nu} = -mu^\mu u^\nu/2$.

It follows from (54) and (56) that the Joukowsky force density exerted by the axionic fluid on a string (of any kind) is given by

$$\bar{f}_{\rm J}^{\mu} = \tfrac{1}{2}\,\bar{\kappa} N^{\mu}{}_{\nu\rho}\mathscr{E}^{\nu\rho} \tag{68}$$

It follows from (55) using (64) and (67) that in the case of a Nambu–Goto string the corresponding dilatonic force density contribution will be obtainable—with the aid of the defining formula (31) for the curvature vector K^μ—in the form

$$\bar{f}_{\rm D}^{\mu} = 2m_{\rm K}^2(\phi K^\mu - \perp^{\mu\nu} \nabla_\nu \phi) \tag{69}$$

Simple though it is, this formula does not seem to have been previously made available in the literature.

It is to be observed that—as needed to avoid overdetermination in the Goto–Nambu case—the force contributions (68) and (69) are both identically orthogonal to the string worldsheet. It is evident that if the dilatonic field were due only to high-frequency radiation from an external source, then the first term on the right in (69) would be relatively negligible, but it will be shown below that this term is not at all negligible for a dilatonic field due to self-interaction.

6. ALLOWANCE FOR REGULARIZED SELF-INTERACTION

In cases where self-interaction is involved, it is commonly convenient to decompose the relevant linear interaction fields—which in the present instance are $B_{\mu\nu}$ and ϕ—into a short-range contribution determined via the relevant Green function by the immediately neighboring source distribution, and a longer range residual contribution that includes allowance for incoming radiation from external sources. More particularly, in the present instance, it will be useful to consider the relevant fields $B_{\mu\nu}$ and ϕ the sums of short-range contributions that will be indicated by a caret and long-range parts that will be indicated by a tilde in the form

$$B_{\mu\nu} = \tilde{B}_{\mu\nu} + \hat{B}_{\mu\nu}, \qquad \phi = \tilde{\phi} + \hat{\phi} \tag{70}$$

This will evidently give rise to corresponding decompositions

$$f_{\mathrm{J}}^{\mu} = \tilde{f}_{\mathrm{J}}^{\mu} + \hat{f}_{\mathrm{J}}^{\mu}, \qquad f_{\mathrm{D}}^{\mu} = \tilde{f}_{\mathrm{D}}^{\mu} + \hat{f}_{\mathrm{D}}^{\mu} \tag{71}$$

for the associated force densities as specified by the general formulas (53), (54) or their Nambu–Goto string specializations (68), (69).

In many contexts the coupling is so weak that the local self-force contributions $\hat{f}_{\mathrm{J}}^{\mu}$ and $\hat{f}_{\mathrm{D}}^{\mu}$ can be neglected. However, in cases for which one needs to take account of the self-induced contributions $\hat{B}_{\mu\nu}$ and $\hat{\phi}$, there will be difficulties arising from the fact that the relevant source fields on the right-hand sides of the field equations (11) and (12) will not be the regular worldsheet-supported tensor fields $\overline{W}^{\mu\nu}$ and $\overline{T}$, but the corresponding four-dimensionally supported distributions $\hat{W}^{\mu\nu}$ and $\hat{T}$ as constructed according to the prescriptions (14) and (15). For sources such as these, the resulting field contributions will diverge in the thin-worldsheet limit.

As in the familiar point-particle case, so also for a string, one can obtain an appropriately regularized result by supposing that (as will be entirely realistic in cases such as that of a cosmic string model for a vortex defect of the vacuum) the underlying physical system one wishes to describe is not quite infinitely thin, but actually has finite spatial extent that can be used to specify an appropriately microscopic "ultraviolet" cutoff length scale, δ_* say. This will be sufficient for regularization in the point-particle case, but in the string case it will also be necessary to introduce a long-range "infrared" cutoff length scale, Δ say, that might represent the macroscopic mean distance between neighboring strings. In the case of a string an an ordinary four-dimensional spacetime background, it can be seen [following the example (Carter, 1997b) of the electromagnetic prototype considered by Witten (1985) in his original discussion of "superconducting" cosmic strings] that the dominant contribution to relevant Green function integrals in the ultraviolet limit will then be proportional to a logarithmic regularization factor of the form

$$\hat{l} = \ln\{\Delta^2/\delta_*^2\} \tag{72}$$

More specifically, the dominant contribution to the axionic self-field arising from the D'Alembertian source equation (11) will be given by

$$\hat{B}_{\mu\nu} = \hat{l}m_J^2\overline{W}_{\mu\nu} \tag{73}$$

with $\overline{W}_{\mu\nu}$ given by (56), while similarly the dominant contribution to the dilatonic self-field arising from the d'Alembertian source equation (12) will be given by

$$\hat{\phi} = \hat{l}m_D^{-2}\overline{T} \tag{74}$$

(If the microscopic axial current source distribution were very different from that of the stress-energy trace source for the dilatonic field, the natural cutoff δ_* that would be most appropriate for the latter might differ somewhat from what would apply to the former, but the effect of such a difference could be considered as a higher order correction that need not be taken into account so long as we are only concerned with the dominant contribution.)

For the purposes of substitution in the force formulas—(54), (55) or their Nambu–Goto string specializations (68), (69)—knowledge just of the regularized self-fields $\hat{B}_{\mu\nu}$ and $\hat{\phi}$ is not immediately sufficient. The problem is that these regularized values are well defined only on the worldsheet, which means that they do not directly provide what is needed for a direct evaluation of the gradients that are required: there is no difficulty for the terms involving just the tangentially projected gradient operator $\overline{\nabla}_\nu$, but it can be seen that there are also contributions from the unprojected gradient operator ∇_ν which is directly meaningful only when acting on fields whose support extends off the worldsheet.

It fortunately turns out that this problem has a very simple general solution, of which particular applications in particular gauges are implicit in much previous work (Dabholkar and Quashnock 1990a, 1990b; Quashnock and Spergel, 1990; Copeland *et al.*, 1990; Kakushadze, 1993; Battye and Shellard, 1995, 1996) and which I formulated explicitly in conveniently covariant and more generally utilizable form in the specific context of the electromagnetic case (Carter, 1997). What one finds— by examining the string worldsheet limit behavior of derivatives of the relevant d'Alembertian Green function—is that the appropriate regularization of the gradients on the string worldsheet is obtainable simply by replacing the ill-defined operator ∇_ν by the corresponding regularized gradient operator

$$\hat{\nabla}_\nu = \overline{\nabla}_\nu + \tfrac{1}{2} K_\nu \tag{75}$$

where K^μ is the worldsheet curvature vector that is defined by the formula (31) and that is expressible, if one is willing to allow oneself to be restricted to the use of conformal worldsheet coordinates, by a more detailed prescription of the form (61). In the explicit application of the formula (75), it is sufficient, in the case of a scalar field φ, to use the simple expression $\overline{\nabla}^\mu \varphi = \gamma^{ij} x^\mu_{,i} \varphi_{,j}$ for the tangentially projected gradient, but for a tensorial field there will also be contributions depending on the background Riemann Christoffel connection $\Gamma_\mu{}^\nu{}_\rho$, which is also involved in the detailed expressions (33) and (61), unless one is using Minkowski coordinates in a flat spacetime background.

Applying this procedure to the axionic Kalb–Ramond 2-form, one sees from (6) and (73) that the corresponding regularized local current 3-form contribution is given by

$$\hat{J}_{\mu\nu\rho} = 3\hat{\nabla}_{[\mu}\, \hat{B}_{\nu\rho]} = \tfrac{1}{2}\, \hat{l}(6\overline{\nabla}_{[\mu} \overline{W}_{\nu\rho]} + 3\ K_{[\mu} \overline{W}_{\nu\rho]}) \tag{76}$$

Taking account of the surface flux conservation law (37), and using the defining relation (31) for the curvature vector K^μ, it can be seen that the axionic force density (54) will be expressible as a worldsheet divergence in the form

$$\hat{f}^\mu_{\mathrm{J}} = -\overline{\nabla}_\nu \hat{T}^{\mu\nu}_{\mathrm{J}} \tag{77}$$

in terms of a regularized local axionic stress-energy tensor given by

$$\hat{T}^{\mu\nu}_{\mathrm{J}} = \hat{B}^{\mu}_{\ \rho} \overline{W}^{\nu\rho} - \tfrac{1}{4}\, \hat{B}_{\rho\sigma} \overline{W}^{\rho\sigma} \eta^{\mu\nu} \tag{78}$$

which can be seen from (56) and (73) to reduce with the aid of (58) to the simple explicit form

$$\hat{T}^{\mu\nu} = -\tfrac{1}{2}\, \hat{l} \bar{\kappa}^2 m_{\mathrm{J}}^2\ \eta^{\mu\nu} \tag{79}$$

which is evidently isotropic with respect to the two-dimensional worldsheet geometry, like the intrinsic stress-energy tensor in the Nambu–Goto case.

When one applies the same procedure to the dilatonic self-force contribution in (54) one finds that it, too, can be formulated as a worldsheet divergence in the analogous form

$$\hat{f}^\mu_{\mathrm{D}} = -\overline{\nabla}_\nu \hat{T}^{\mu\nu}_{\mathrm{D}} \tag{80}$$

in terms of a regularized local dilatonic stress-energy tensor given by

$$\hat{T}_{\mathrm{D}}^{\mu\nu} = \hat{\phi}(\overline{T}^{\mu\nu} - \tfrac{1}{4}\overline{T}\eta^{\mu\nu} + \overline{\mathscr{C}}^{\mu\nu}) \tag{81}$$

More specifically, in the particular case of a Nambu–Goto-type string, as characterized by (64) and (67), it can be seen that the dominant self-force contribution obtained from (69) is given by

$$\hat{f}_{\mathrm{D}}^{\mu} = m_{\mathrm{K}}^{2}\hat{\phi}K^{\mu} \tag{82}$$

[which is the exact opposite of what would be obtained if the first term on the right-hand side of (69) were omitted] and that the corresponding dilatonic stress-energy contribution reduce to the form

$$\hat{T}_{\mathrm{D}}^{\mu\nu} = 2\hat{l}m_{\mathrm{K}}^{4}m_{\mathrm{D}}^{-2}\eta^{\mu\nu} \tag{83}$$

Comparing (83) with (79), it can be seen that [in contradiction with previous assertions to the effect that it would vanish (Copeland *et al.*, 1990) or even that it would augment the corresponding axionic contribution (Dabholkar and Harvey, 1989)], this dilatonic contribution must always be oppositely directed to the corresponding axionic contribution. More particularly, it can be seen that these dominant local axionic and dilatonic self-interaction contributions will exactly cancel each other out if the relevant coupling constants are related by the condition

$$2m_{\mathrm{K}}^{2} = \bar{\kappa}m_{\mathrm{D}}m_{\mathrm{J}} \tag{84}$$

which will in fact be satisfied automatically in the special case (63) envisaged by Dabholkar and Harvey (1989) [whose faulty analysis gave a condition that was somewhat different from (84)—but that also happened to be satisfied in the special case (63) they were considering, and that thereby provided a spurious verification of their reasoning].

The mutual cancellation subject to (84) of the dominant axionic and dilatonic self-interactions for a Nambu–Goto string in a four-dimensional background has already been confirmed by Buonanno and Damour (1998) using an entirely different approach formulated in terms of an effective action. The fact that—as in the previously investigated electromagnetic (Carter, 1997b) and gravitational (Carter and Battye, 1998; Carter 1999) cases—the result of the present approach based on direct evaluation of the self-force is in full agreement with the result of the approach based on the use of an effective action provides a check on the validity of the latter approach, whose credibility had previously been questioned (Copeland *et al.*, 1990). The complete consistency between the two kinds of approach has already been made clear for the electromagnetic (Carter, 1997) and gravitational (Carter, 1999) cases, and will be made clear for the axionic and dilatonic cases dilatonic in the next section, where the relevant effective action contributions will be explicitly derived.

7. ACTION RENORMALIZATION AND CONCLUSIONS

The fact that the dominant local force contributions are expressible as divergences of the form (79) and (80) is what makes it possible to describe the the result of this regularization as a "renormalization": it implies that these self-force contributions can be absorbed into the left-hand side of the basic force balance equation by a renormalization whereby the original "bare" stress-momentum-energy density tensor $\overline{T}^{\mu\nu}$ undergoes a replacement $\overline{T}^{\mu\nu} \mapsto \tilde{T}^{\mu\nu}$ in which the "dressed" stress-momentum-energy tensor is given by

$$\tilde{T}^{\mu\nu} = \overline{T}^{\mu\nu} + \hat{T}^{\mu\nu} \tag{85}$$

with

$$\hat{T}^{\mu\nu} = \hat{T}_{\mathrm{J}}^{\mu\nu} + \hat{T}_{\mathrm{D}}^{\mu\nu} \tag{86}$$

The basic force balance equation (51) can thereby be rewritten in the equivalent form

$$\overline{\nabla}_{\nu}\tilde{T}^{\mu\nu} = \tilde{f}^{\mu} \tag{87}$$

in which the force density on the right consists just of well-behaved long-range radiation contributions as given by the sum

$$\tilde{f}^{\mu} = \tilde{f}_{\mathrm{J}}^{\mu} + \tilde{f}_{\mathrm{D}}^{\mu} \tag{88}$$

in which each of the terms is entirely regular.

What will be shown in this final section is that the renormalized stress-energy tensor $\tilde{T}^{\mu\nu}$ is derivable, by a prescription of the standard form (20), from a correspondingly renormalized action in which the original Lagrangian master function $\mathcal{L}_{\mathrm{ma}}$ is replaced by an appropriately renormalized master function $\tilde{\mathcal{L}}_{\mathrm{ma}}$.

In order to incorporate the effects of self-interaction, as described by the renormalized force balance equation (87), it can be verified that all one needs to do is to replace $\mathcal{I}_{\mathrm{ma}}$ by a corresponding renormalized action

$$\tilde{\mathcal{I}}_{\mathrm{ma}} = \int \tilde{\mathcal{L}}_{\mathrm{ma}} \|\gamma\|^{1/2}\, d^2\sigma \tag{89}$$

with

$$\tilde{\mathcal{L}}_{\mathrm{ma}} = \tilde{\mathcal{L}} + \tfrac{1}{2}\,\tilde{B}_{\mu\nu}\overline{W}^{\mu\nu} + \tilde{\phi}\overline{T} \tag{90}$$

where the renormalized master function is given simply by

$$\tilde{\mathcal{L}} = \overline{\mathcal{L}} + \hat{\Lambda}_{\mathrm{J}} + \hat{\Lambda}_{\mathrm{D}} \tag{91}$$

in which the axionic self-coupling contribution is given by

$$\hat{\Lambda}_{\rm J} = \tfrac{1}{4}\,\hat{B}_{\mu\nu}\overline{W}^{\mu\nu} \tag{92}$$

which works out simply to be a constant,

$$\hat{\Lambda}_{\rm J} = -\tfrac{1}{2}\,\hat{l}\bar{\kappa}^2 m_{\rm J}^2 \tag{93}$$

in which it is to be recalled that $\hat{l}$ is the logarithmic factor given by (72). The corresponding dilatonic self-coupling contribution (which has not been evaluated for a general string model before) is obtained as

$$\hat{\Lambda}_{\rm D} = \tfrac{1}{2}\,\hat{\phi}\overline{T} = \tfrac{1}{2}\,\hat{l}m_{\rm D}^{-2}\overline{T}^2 \tag{94}$$

It is to be observed that in terms of the string energy density $\mathcal{U}$ and tension $\mathcal{T}$ (as conventionally defined to be the eigenvalues of $-\overline{T}^{\mu\nu}$) the trace in the preceding formula is given by $\overline{T} = -(\mathcal{U} + \mathcal{T})$, so the dilatonic self-interaction contribution is proportional to the $(\mathcal{U} + \mathcal{T})^2$. This is to be contrasted with the case of the corresponding gravitational self-interaction contribution, which has been shown (Carter, 1999) to be proportional to $(\mathcal{U} - \mathcal{T})^2$. For a Nambu–Goto string, as characterized by $\mathcal{U} = \mathcal{T} = m_{\rm K}^2$, this gravitational contribution will simply vanish, while the dilatonic contribution (94) will just have the constant value given by

$$\hat{\Lambda}_{\rm D} = 2\hat{l}m_{\rm K}^4 m_{\rm D}^{-2} \tag{95}$$

The self-interaction contributions (93) and (95) are in perfect agreement with those already obtained by the effective action approach developed by Buonanno and Damour (1998), whose results were more general than those provided here insofar as they were not limited to a four-dimensional background, though on the other hand they were more restricted insofar as they considered only strings of Nambu–Goto type.

This completes the demonstration that, as shown for electromagnetic (Carter, 1997b) and gravitational (Carter, 1999) interactions, and contrary to what was previously alleged (Copeland *et al.*, 1990), so also for the axionic and dilatonic contributions, the treatment of the divergent self-interaction contribution by an approach developed directly in terms of effective action is entirely consistent with the treatment based on the detailed analysis of the corresponding force contributions, as given in the axionic case by (68) and in the dilatonic case by the general formula (55) or its Nambu–Goto specialization (69).

The discrepancies that arose in earlier work were due to the use of unreliable short-cut methods [largely motivated by the unavailability of the more efficient methods of geometrical analysis (Carter, 1997a) that have since been developed], which lead to the omission of some of the (less easily calculable) terms in the relevant formulas. In particular, the sign error in the original estimate of the net dilatonic contribution by Dabholkar and Harvey

(1989) can be accounted for as being due to omission of the contribution provided by the first term on the right in (69). As was seen for the corresponding forces in the previous section, the axionic contribution (93) will evidently be exactly canceled by the dilatonic contribution (94) when the special condition (84) is satisfied. Like the self-cancellation of the gravitational contribution in the Nambu–Goto case (Carter and Battye, 1998; Carter, 1999), this mutual cancellation of the corresponding axionic and dilatonic contributions is a special feature of four-dimensional space-time: it has been shown by the work of Buonanno and Damour (1998) that it does not carry over to Nambu–Goto strings in backgrounds of higher dimension.

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